Exercise 14

Use the method of undetermined coefficients to find the general solution for the following second order ODEs:

$$u'' + u = 3$$

Solution

This is an inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$u = u_c + u_p$$

The complementary solution is the solution to the associated homogeneous equation,

$$u_c'' + u_c = 0.$$

This is a linear ODE with constant coefficients, so the solution will be of the form $u_c = e^{rx}$.

$$u_c = e^{rx} \quad \to \quad u_c' = re^{rx} \quad \to \quad u_c'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} + e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 + 1 = 0$$

Factor the left side.

$$(r + i)(r - i) = 0$$

$r = -i$ or $r = i$, so the complementary solution is

$$u_c(x) = C_1 e^{-ix} + C_2 e^{ix}.$$

We can write this in terms of sine and cosine by Euler's formula.

$$u_c(x) = A \cos x + B \sin x$$

Now we turn our attention to the particular solution. Because the inhomogeneous term, 3, is a constant, the particular solution should be chosen so that the higher derivatives vanish but the smallest derivative remains, i.e. $u_p = C$. Plugging this form into the ODE yields $C = 3$. Thus, $u_p = 3$. Therefore, the general solution to the ODE is

$$u(x) = A \cos x + B \sin x + 3.$$

We can check that this is the solution. The first and second derivatives are

$$u' = -A \sin x + B \cos x$$
$$u'' = -A \cos x - B \sin x.$$

Hence,

$$u'' + u = -\cancel{A \cos x} - \cancel{B \sin x} + \cancel{A \cos x} + \cancel{B \sin x} + 3 = 3,$$

which means this is the correct solution.